rate with atomic number is appreciably greater in the present data than in that previously reported. This result would appear to be associated with the above noted fact that the increase in counting rate due to showers from the heavier elements, as measured by our arrangement of counters, is not a linear function of the thickness of producing material. This follows because our earlier data were multiplied by the atomic weight to obtain the relative frequency of showers per unit atomic density. Such a procedure would only be accurate if the shower frequency *vs.* thickness curve were linear in each case.

The recent theoretical results of Carlson and Oppenheimer[7] and of Bhabha and Heitler[8] have indicated that the multiplication theory of showers is capable of accounting, in a rough way, for the showers due to the softer component of the general cosmic-radiation. It would appear that such a theory, if correctly applied, can account for a variation of shower production with a second or relatively small power of the atomic number. It is apparently not necessary to assume from such a dependence on atomic number that the showers have their origin in a single act. Any detailed analysis of results such as those herein reported would involve a number of complicating factors. An important one of these factors is the difference in angular spread of the showers from various materials, and its effect on the efficiency of the counting apparatus. Although such a complete analysis is impossible, it appears desirable to record these observations, the results of which are consistent with those of our previous data obtained under identical experimental conditions.

[7] J. F. Carlson and J. R. Oppenheimer, Phys. Rev. **51**, 220 (1937).

[8] H. J. Bhabha and W. Heitler, Proc. Roy. Soc. **159A**, 432 (1937).

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On Fluctuation Phenomena in the Passage of High Energy Electrons through Lead

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It now seems reasonable on both theoretical and experimental grounds to suppose that the formulae of the present radiation theory are valid in the cosmic-ray energy range. The work of Carlson and Oppenheimer and of Bhabha and Heitler has shown that this assumption is capable of accounting for many of the observed features of cosmic-ray absorption and of shower production. These writers concern themselves principally with the mean behavior of a group of electrons and photons moving through matter. Since the fluctuations around this mean behavior are large and for some purposes very important, it is desirable to investigate their nature, even though this involves a loss of accuracy in dealing with other aspects of the situation. In this paper we consider two fluctuation problems: (1) The fluctuations in size of showers produced by single electrons or photons: In dealing with this problem we take the energetic relations into account only very roughly. (2) Fluctuations in energy loss of electrons: The possible production of secondaries is disregarded. The inadequacies in treatment have for both problems the consequence that the results are applicable only to thin layers of heavy substances. The first problem is discussed in Section II. The conclusion is that the distribution in shower sizes should be essentially of the type $P(n;\langle n\rangle) = (\langle n\rangle)^{-1}\{1-(\langle n\rangle)^{-1}\}^{n-1}$, where $\langle n\rangle$ is the mean number; but that under ordinary experimental conditions the number of very small showers should be rather greater than indicated by this law. The results account for two observed phenomena which might at first sight be taken as forming serious objections to the multiplicative hypothesis: First, the occasionally observed production of large showers ($\sim$20 or 30 particles) from small thicknesses ($\sim$1 cm) of lead; and second, the appearance which many of the larger showers present of having originated at a single point near the bottom of the lead. In Section III we deal with the second problem, with the purpose of providing a way to use energy loss measurements to provide a more detailed check on the theoretical formulae. A method is given for constructing energy loss distribution curves corresponding to any assumed form of the *Bremsstrahlung* spectrum. Also a solution is outlined for the problem of using accurate and detailed information on energy losses to compute an empirical spectrum curve.

I. Introduction

SOME months ago, Anderson and Neddermeyer[1] published the results of a new series of measurements of the energy losses incurred by cosmic-ray electrons in passing through a thin sheet of lead. These results indicate that the formula for the loss of energy by radiative collisions (*Bremsstrahlung*) calculated[2] according to the present radiation theory, is probably obeyed up to primary energies of some hundreds of millions of volts; and on theoretical grounds[3] it is reasonable to suppose that it holds to much greater energies. There is also a strong presumption that the corresponding formula for absorption of photons by pair production[2] is valid in the range of energies of the cosmic rays.

These assumptions make the picture of what happens in the passage of cosmic-ray electrons and photons through matter a quite complicated one. The dominant feature of the process is the rapid decrease in the amount of energy carried by any individual particle, together with a rapid increase in the number of fast electrons and high energy photons present. The events through which these results come about are of an altogether accidental and random character, so that the behavior of such a "multiplicative shower" in any given case may differ widely from the average behavior. The main points to be taken into account in discussing such a situation are accordingly:

(a) Loss of energy by individual particles.
(b) Production of secondary particles.
(c) Fluctuations.

In dealing with so complicated a matter it has been necessary not only to simplify the calculations by using suitable approximations to the theoretical formulae involved, but also to confine attention to certain aspects only. The calculations of Carlson and Oppenheimer[4] and of Bhabha and Heitler,[5] which have shown that this general picture can account for many of the observed facts of cosmic-ray absorption and shower production, were concerned essentially with the treatment of points (a) and (b),[6] the fluctuations being given only slight consideration. The purpose of the present paper is to obtain further evidence about the fluctuations, which is done at the expense of omitting from consideration one or the other of the first two points. Though this may appear a very crude procedure, it is possible to assign physical conditions for which the results should be significant; and the fluctuations are found to be so great that information about them seems comparable in importance with information about the other aspects of the situation.

In Section II the question of fluctuations in the number of particles present in a shower is taken up; that is, the problem is centered about points (b) and (c), point (a) being omitted. Such considerations can be regarded as pertinent only for the case of showers produced in a *small* thickness of a *heavy* material, such as lead: for larger thicknesses the loss of particles by stopping becomes comparable with their production; and in light materials this will be so even at the smallest thicknesses which are capable of occasioning an appreciable production of secondaries. In spite of their limited application, these calculations are of importance because they show to what extent the theory can account for the occasionally observed production of a large shower from a fairly thin layer of lead, which might at first sight seem to be a grave objection against the multiplication hypothesis. The results indicate that the fluctuations are very large indeed, and that the occasional production of showers with several times the expected mean number of particles is quite in accordance with the theory.

In Section III we consider the fluctuations in

[1] C. D. Anderson and S. H. Neddermeyer, Phys. Rev. **50**, 263 (1936).

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[3] C. F. v. Weizsäcker, Zeits. f. Physik **88**, 612 (1934); E. J. Williams, Phys. Rev. **45**, 729 (1934); D. Kgl. Danske Widensk. Selskab., Math-fys. Meddelelser **13**, 4 (1935).

[4] J. F. Carlson and J. R. Oppenheimer, Phys. Rev. **51**, 220 (1937).

[5] H. J. Bhabha and W. Heitler, Proc. Roy. Soc. **A159**, 432 (1937).

[6] In treating this combination, it is of course necessary to require that the energy lost by the electrons through *Bremsstrahlung* appears as the energy of new light quanta, and that the pairs produced receive the energy of the light-quanta absorbed. The fluctuations in number of particles are neglected in both calculations, the formulae used being concerned only with mean numbers; some account is taken of fluctuations in radiative energy loss, by the use of rough approximations.

radiative energy loss of an electron, leaving out of account the production of secondary electrons. (Points (a) and (c), point (b) omitted.) Here again the results are of interest principally only for a small thickness of a heavy substance: in light substances ionization loss, which cannot be taken into account with any accuracy, is comparable in importance with the radiative loss; and in thicker layers of lead the situation is inevitably obscured by the presence of secondaries. We are thus concerned not with the problem of straggling in range, in which connection such calculations were originally made by Bethe and Heitler, but rather with putting the theory into shape for comparison with energy loss experiments made with thin sheets of lead. The most practical procedure for use with a limited amount of data seems to be to compute the distribution curves for energy losses corresponding to various assumed forms of the *Bremsstrahlung* spectrum; a general method of doing this is developed. At the end of the section a procedure is outlined for solving the converse problem: Given a very large amount of highly accurate energy loss data, to compute the corresponding form of the spectrum.

II. Distribution in Size of Multiplicative Showers

In the calculations of this section we leave out of account the questions of distribution of the particles in energy, and of energy loss; some indication of the probable effects of these considerations on the results will be given at the end of the section. In order for such a treatment to be approximately valid, it is necessary that the primary energy be large compared to the total ionization loss which would be suffered by all the particles of the largest shower to be considered, in traversing a thickness comparable with that in question. This condition is satisfied for primary energies of some hundreds of millions of volts and thicknesses ~ 1 cm Pb, if one wishes to consider showers ranging up to twenty or thirty particles.

Following the notation of Carlson and Oppenheimer,[4] we measure thickness in terms of a variable t, such[7] that $t=1$ corresponds to about 0.5 cm Pb. Then the probability that in traversing thickness dt a high energy electron emits a photon of energy comparable with its own is, very roughly, just dt. The probability that in a thickness dt a high energy photon will produce a positron-electron pair is σdt, with σ equal to about two-thirds.

[7] The definition of the variable t is made more precise at the beginning of Section III.

We solve first a decidedly oversimplified problem, which nevertheless serves to indicate the essential nature of the desired distribution:

Problem A: The probability that in traversing thickness dt one particle is converted into two is just dt. If one particle enters a sheet of thickness t, what is the probability $P(n;t)$ that n particles will emerge?

The probabilities in question satisfy the differential equations

$$\left.\begin{aligned} &(d/dt)P(1;t) = -P(1;t)\\ &(d/dt)P(2;t) = -2P(2;t)+P(1;t)\\ &\cdots\cdots\cdots\cdots\\ &(d/dt)P(n;t) = -nP(n;t)\\ &\qquad\qquad +(n-1)P(n-1;t)\\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{1}$$

with the boundary conditions

$$P(1;0)=1; \quad P(n;0)=0, \quad n>1. \tag{2}$$

By successive integration and application of the boundary conditions one obtains

$$P(1;t)=e^{-t}; \;\; P(2;t)=e^{-t}(1-e^{-t}); \;\; \cdots; \tag{3}$$

and it is readily established by mathematical induction that

$$P(n;t)=e^{-t}(1-e^{-t})^{n-1}. \tag{4}$$

The *most probable* number of particles emerging is *one;* the *mean* number is $\langle n\rangle = e^t$. Thus (4) can be written

$$P(n;t)=(\langle n\rangle)^{-1}\{1-(\langle n\rangle)^{-1}\}^{n-1}. \tag{4'}$$

We want the solution to a somewhat different problem:

Problem B: The probability that in dt an electron produces a photon is dt, the electron itself continuing its course; the probability that in dt a photon produces a pair of electrons is σdt, the photon being absorbed. If one electron enters a sheet of thickness t, what is the probability $Q(n;t)$ that n electrons will emerge?

Defining $P(n;m;t)$ as the probability that n

electrons and m photons emerge, one has the differential equations

$$\left.\begin{aligned}
&(d/dt)P(1;0;t)=-P(1;0;t)\\
&(d/dt)P(1;1;t)=-(1+\sigma)P(1;1;t)\\
&\qquad\qquad\qquad\qquad +P(1;0;t)\\
&(d/dt)P(1;2;t)=-(1+2\sigma)P(1;2;t)\\
&\qquad\qquad\qquad\qquad +P(1;1;t)\\
&(d/dt)P(3;0;t)=-3P(3;0;t)\\
&\qquad\qquad\qquad\qquad +\sigma P(1;1;t)\\
&\cdots\cdots\cdots\cdots\\
&(d/dt)P(n;m;t)=-(n+m\sigma)P(n;m;t)\\
&\qquad\qquad +nP(n;m-1;t)\\
&\qquad\qquad +(m+1)\sigma P(n-2;m+1;t)\\
&\cdots\cdots\cdots\cdots
\end{aligned}\right\}\quad(5)$$

with the boundary conditions

$$P(1;0;0)=1;\quad \text{all other } P(n;m;0)=0. \quad (6)$$

After solving for $P(n;m;t)$, one has

$$Q(n;t)=\sum_{m=0}^{\infty}P(n;m;t). \quad (7)$$

The equations can be solved successively like the first set, without difficulty; but the formulae obtained rapidly become very long, and could scarcely be interpreted except by numerical computation. To carry this through for large values of n would be prohibitively laborious. We shall see that the behavior of $Q(n;t)$ can be obtained more easily.

From (5) and (6) one readily finds for $n=1$:

$$P(1;0;t)=e^{-t};\, P(1;1;t)=e^{-t}\sigma^{-1}(1-e^{-\sigma t});\\ \cdots;\, P(1;m;t)=(e^{-t}/\sigma^m m!)(1-e^{-\sigma t})^m \quad (8)$$

and on summing,

$$Q(1;t)=e^{-t+[1-\exp(-\sigma t)]/\sigma}. \quad (9)$$

The mean number of photons accompanying the primary if no secondary electrons have been produced is

$$\langle m_1\rangle=\{Q(1;t)\}^{-1}\sum_m mP(1;m;t)\\ =\sigma^{-1}(1-e^{-\sigma t}). \quad (10)$$

Let us also calculate $\langle m_1'\rangle$, the mean number of photons accompanying a primary which has just produced its first pair of secondary electrons. Keeping in mind that the probability of producing this secondary pair is proportional to the number of photons present, we have

$$\langle m_1'\rangle=\{\sum_m mP(1;m;t)\}^{-1}\sum_m m(m-1)P(1;m;t)\\ =\sigma^{-1}(1-e^{-\sigma t})=\langle m_1\rangle. \quad (11)$$

Thus the mean number of photons belonging to a given electron depends only on the distance the electron has traversed, and not on how many secondaries have been produced by its photons. This makes it possible to formulate the conditions of Problem B in such a way that no explicit reference is made to the photons:

Problem B (restatement): The probability that in dt a given electron will produce a pair is $(1-e^{-\sigma t'})dt$, where t' is the distance the electron has traversed since it was produced or entered the lead. If one electron enters a sheet of thickness t, what is the probability $Q(n;t)$ that n electrons will emerge?

Besides the trivial difference that the secondaries are produced in pairs instead of singly, this formulation differs from Problem A in that an electron's capacity for producing secondaries is not constant, but is zero at the beginning of the electron's path and approaches only asymptotically the constant value assigned to it in Problem A. This dependence on the electron's past history makes the problem much more difficult; it cannot be formulated in terms of ordinary differential equations involving the $Q(n;t)$.[8]

We can use the new formulation to get an expression for $Q(3;t)$. The probability that a secondary pair was produced between t' and $t'+dt'$ and that no other pair was produced is the product of three factors:

(1) The probability that the primary did not produce a pair before reaching t' or after passing $t'+dt'$. Since dt' is infinitesimal, and since the probability of the primary's producing a pair is unaffected by its having already done so, this factor is just $Q(1;t)$.

(2) The probability that the primary did

[8] By introducing the variable $s=\Sigma(1-e^{-\sigma t'})$ and defining $Q(n;s;t)ds$ as the probability that at t there are n electrons and s lies in ds, we can derive the *partial* differential equations

$$\{(\partial/\partial t)+\sigma(n-s)(\partial/\partial s)+(s-\sigma)\}Q(n;s;t)=sQ(n-2;s;t).$$

They seem to be of no particular help in attacking the problem.

produce a pair between t' and $t'+dt'$. This is of course $(1-e^{-\sigma t'})dt'$.

(3) The probability that neither secondary produced a pair in going from t' to t. This is equal to $\{Q(1;t-t')\}^2$. Then, by (9):

$$Q(3;t)=Q(1;t)\int_0^t(1-e^{-\sigma t'})dt'\cdot\{Q(1;t-t')\}^2$$

$$=Q(1;t)e^{2/\sigma}\int_0^t(1-e^{-\sigma t'})dt'e^{-2(t-t')}\cdot e^{-(2/\sigma)\exp[-\sigma(t-t')]}.$$

Setting $(2/\sigma)e^{-\sigma(t-t')}=z$, we have

$$Q(3;t)=\tfrac{1}{2}Q(1;t)e^{2/\sigma}\int_{(2/\sigma)e^{-\sigma t}}^{(2/\sigma)}e^{-z}dz\cdot(\sigma z/2)^{(2/\sigma)-1}(1-(2/\sigma)e^{-\sigma t}z^{-1}). \quad (12)$$

Thus $Q(3;t)$ is expressed in terms of incomplete gamma-functions. For certain rational values of σ—those which make $(2/\sigma)$ an integer—exponentials suffice. For example, one finds that for $\sigma=1$,

$$Q(3;t)=\tfrac{1}{4}Q(1;t)[e^{2t}\{Q(1;t)\}^2+2e^{-t}-3] \quad (12a)$$

and for $\sigma=\frac{2}{3}$,

$$Q(3;t)=(1/18)Q(1;t)[(2+3e^{-2t/3})e^{2t}\cdot\{Q(1;t)\}^2+12\ e^{-2t/3}-17]. \quad (12b)$$

By a similar procedure one can obtain the expressions for $Q(5;t)$, $Q(7;t)$, etc., in terms of multiple integrals; and when $(2/\sigma)$ is an integer, the integrations can be performed in terms of exponentials. This gives a more practicable method for evaluating the $Q(n;t)$ than is provided by Eqs. (5), (6), and (7); but the formulae still increase rapidly in complexity with increasing n, and would require extended numerical computations for their interpretation.

In the case $\sigma=1$ a special method is available for the numerical evaluation of the $Q(n;t)$, and the results suffice to suggest what is almost certainly the form of the answer for other values of σ also. This method depends on the fact that when $\sigma=1$ Problem B differs from Problem A only by our making a distinction between the two kinds of particles (electrons and photons). We then have

TABLE I. *Values of the* q_{nm}.

$n+m$ \ n	1	3	5	7
1	1.00000			
2	1.00000			
3	.50000	.50000		
4	.16667	.83333		
5	.04167	.75000	.20833	
6	.00833	.48333	.50833	
7	.00139	.24861	.66528	.08472
—	—	—	—	—

$$Q(n;t)=\sum_{m=0}^{\infty}q_{nm}P(n+m;t), \quad (13)$$

where the $P(l;t)$ are given by (4), and q_{nm} is the probability that if there are $n+m$ particles, n are electrons. The q_{nm}'s satisfy the relations

$$q_{nm}=nq_{n,\,m-1}/(n+m-1)+(m+1)q_{n-2,\,m+1}/(n+m-1) \quad (14)$$

with the boundary condition

$$q_{10}=1,\quad q_{01}=0. \quad (15)$$

Table I shows the values of a few of the q_{nm}. Its manner of construction by the use of (14) is obvious. Each row of entries is computed from the preceding row, and a check is provided by the fact that the sum across each row is unity. Such a table was prepared extending to $n+m=50$, and was used together with (4) and (13) to compute the $Q(n;t)$ for $\sigma=1$.

The results are summarized in Table II. Only

TABLE II. ($\sigma=1$.)

n	$Q(n;1)$	$\rho_n(1)$	$Q(n;1.5)$	$\rho_n(1.5)$	$Q(n;2)$	$\rho_n(2)$	$Q(n;2.5)$	$\rho_n(2.5)$
1	0.69220	—	0.48523	—	0.32132	—	0.20555	—
3	.22083	.3190	.26389	.5439	.23356	.7269	.17649	.8586
5	.06257	.2833	.12911	.4893	.15388	.6588	.13796	.7817
7	.01755	.2805	.06268	.4855	.10071	.6545	.10719	.7770
—	—	—	—	—	—	—	—	—
Limit	—	.2802	—	.4852	—	.6542	—	.7767

a few entries have to be given, for the reason that the quantities

$$\rho_n(t)=Q(n;t)/Q(n-2;t) \quad (16)$$

very rapidly approach a constant limiting value. The distribution for Problem B, $\sigma=1$, is thus of almost exactly the same type as for Problem A.

It is somewhat embarrassing to have such a simple behavior turn up, without being able to give an abstract proof that it should. The writer has devoted considerable time and thought to various attempts to obtain such a proof, but without success. At any rate, for $\sigma=1$ the simple behavior does appear, and the assumption that it holds for other values of σ is very plausible: it is owing to the qualitative difference between Problem B—cf. especially the second formulation—and Problem A, and not to the particular value of σ, that one would have expected any decided departure from the simple type of distribution.

The validity of this assumption for $\sigma=\frac{2}{3}$ was tested by the following procedure: $Q(1;t)$ and $Q(3;t)$ were computed from Eqs. (9) and (12b). The rest of the distribution was approximated by setting

$$Q^*(3+2l;t)=\rho^l Q(3;t) \tag{17}$$

and determining ρ from the requirement that the sum of all the probabilities be unity:

$$\begin{aligned} &Q(1;t)+Q(3;t)/(1-\rho)=1;\\ &\rho=1-Q(3;t)/\{1-Q(1;t)\}. \end{aligned} \tag{18}$$

This approximate distribution was then used to calculate approximate values of $\langle n\rangle$, the mean number of electrons emerging, and $\langle n^2\rangle$, the mean square. One obtains

$$\langle n\rangle \cong 1+2Q(3;t)/(1-\rho)^2, \tag{19}$$

$$\begin{aligned} \langle n^2\rangle \cong\, &Q(1;t)+\{Q(3;t)/(1-\rho)\}\\ &\times[9+16\rho/(1-\rho)+8\{\rho/(1-\rho)\}^2]. \end{aligned} \tag{20}$$

These values were then checked against the exact values of these quantities, which satisfy the equations:

$$(d/dt)\langle n\rangle=2\sigma\langle m\rangle;\ (d/dt)\langle m\rangle=\langle n\rangle-\sigma\langle m\rangle, \tag{21}$$

$$\left.\begin{aligned} &(d/dt)\langle n^2\rangle=4\sigma\langle nm\rangle+4\sigma\langle m\rangle\\ &(d/dt)\langle nm\rangle=\langle n^2\rangle-\sigma\langle nm\rangle+2\sigma\langle m^2\rangle-2\sigma\langle m\rangle\\ &(d/dt)\langle m^2\rangle=2\langle nm\rangle-2\sigma\langle m^2\rangle+\langle n\rangle+\sigma\langle m\rangle \end{aligned}\right\} \tag{22}$$

and the boundary conditions

$$\langle n\rangle=\langle n^2\rangle=1,\ \langle m\rangle=\langle nm\rangle=\langle m^2\rangle=0,\ \text{at } t=0. \tag{23}$$

Here as usual m denotes number of photons. From (21), (22), (23) one finds, for $\sigma=\frac{2}{3}$:

$$\langle n\rangle=0.63867e^{0.86852t}+0.36133e^{-1.53518t}; \tag{24}$$

$$\begin{aligned} \langle n^2\rangle=\,&0.79065e^{1.73704t}-0.15073e^{0.86852t}\\ &-0.46156e^{-0.66667t}+0.63351e^{-1.53518t}\\ &+0.18813e^{-3.07036t}. \end{aligned} \tag{25}$$

The results of the computation are shown in Table III. Values calculated from (19) and (20) are indicated by the sign $\cong$, the exact values from (24) and (25) by $=$. It is seen that the agreement is good. The increasing discrepancy for larger values of t is presumably due to the increasing importance of slight differences between ρ and the actual limiting ratio; obviously their effect would be most pronounced in the case of $\langle n^2\rangle$. To check this supposition, an alternative calculation was made for $t=2.5$: The approximation (17) was replaced by

$$Q^*(5+2l;t)=\rho^l Q^*(5;t), \tag{17'}$$

and $Q^*(5;t)$ and ρ were determined by the requirements that the sum of all probabilities be unity and that $\langle n\rangle$ receive its correct value as given by (24). The results were:

$$Q^*(5;t)=.15766,\quad \rho=.68426,\quad \langle n^2\rangle\cong 59.506.$$

The agreement with the exact value of $\langle n^2\rangle$ is now very good, and the difference between the ρ of Table III and the new ρ is seen to be of the same order of magnitude as those between ρ_5 and the limiting ρ in Table II.

Photon-produced showers: Problem B'.—Keeping the essential postulates of Problem B fixed, we may ask: *If one photon enters a sheet of*

TABLE III. ($\sigma=\frac{2}{3}$.)

t	1.0	1.5	2.0	2.5
$Q(1;t)$	0.76328	0.57590	0.40845	0.27712
$Q(3;t)$	.18664	.25882	.26284	.22354
ρ	.21240	.38972	.55568	.69076
$\langle n\rangle\cong$	1.6005	2.3898	.36628	5.6753
$\langle n\rangle=$	1.6000	2.3861	.36447	5.6087
$\langle n^2\rangle\cong$	4.0564	10.110	24.971	61.473
$\langle n^2\rangle=$	4.0403	10.045	24.565	59.411

TABLE IV. ($\sigma=1$.)

n	$R(n;1)$	$\rho_n(1)$	$R(n;1.5)$	$\rho_n(1.5)$	$R(n;2)$	$\rho_n(2)$	$R(n;2.5)$	$\rho_n(2.5)$
0	0.36788	—	0.22313	—	0.13534	—	0.08208	—
2	.46728	1.2702	.41604	1.8646	.31377	2.3185	.21631	2.6354
4	.11876	.2542	.18600	.4471	.19079	.6081	.15693	.7255
6	.03316	.2792	.09000	.4839	.12453	.6527	.12164	.7751
8	.00929	.2801	.04366	.4851	.08146	.6541	.09447	.7766
—	—	—	—	—	—	—	—	—
Limit	—	.2802	—	.4852	—	.6542	—	.7767

TABLE V. ($\sigma=\frac{2}{3}$, *photon-produced showers.*)

t	1.0	1.5	2.0	2.5
$\langle n\rangle \cong$	1.2024	1.9844	3.1166	4.8086
$\langle n\rangle =$	1.2026	1.9856	3.1252	4.8526
$\langle n^2\rangle \cong$	3.5609	8.624	20.612	48.642
$\langle n^2\rangle =$	3.5628	8.647	20.826	50.077

lead t units thick, what is the probability $R(n;t)$ that n electrons emerge?

Beginning with the case $\sigma=1$, we have

$$R(n;t)=\sum_{m=0}^{\infty} r_{nm}P(n+m;t), \qquad (13')$$

where the r_{nm} obey the same relations (14) as do the q_{nm}, but with the boundary conditions

$$r_{10}(0)=0, \quad r_{01}(0)=1. \qquad (15')$$

Obviously $R(0;t)=e^{-t}$: this follows also because (14) and (15′) give $r_{0m}=0$ for $m>1$. By constructing a table of the r_{nm} similar to that of the q_{nm} (cf. Table I), the results given in Table IV were computed. (Here $\rho_n=R(n;t)/R(n-2,t)$.) It is seen that the sequences, apart from their first terms $R(0;t)$, approach very rapidly to a geometric progression just as do the $Q(n;t)$; and the limiting values of the ratios are the same as for the $Q(n;t)$.

By a procedure analogous to that leading to (12b), one finds that, for $\sigma=\frac{2}{3}$:

$$\left.\begin{aligned} R(2;t)&=(1/9)R(0;t)[(1+3e^{-2t/3})e^{2t}\cdot\{Q(1;t)\}^2-4] \\ R(4;t)&=(1/108)R(0;t)[(1+6e^{-2t/3}+6e^{-4t/3})e^{4t}\{Q(1;t)\}^4-12(1+3e^{-2t/3}-4e^{-4t/3})e^{2t}\{Q(1;t)\}^2-13]\end{aligned}\right\} \qquad (26)$$

where $Q(1;t)$ is as defined by (9). If we approximate the rest of the distribution by setting

$$R^*(4+2l;t)=\rho^l R(4;t), \qquad (27)$$

where ρ is determined by requiring that the sum of all probabilities be unity, we can compute approximate values for $\langle n\rangle$ and $\langle n^2\rangle$. Exact values can be computed by integrating (21) and (22) under the boundary conditions

$$\langle n\rangle=\langle n^2\rangle=\langle nm\rangle=0,\ \langle m\rangle=\langle m^2\rangle=1, \text{ at } t=0. \qquad (23')$$

The comparison of the results is given in Table V. It is seen that the agreement is good.

These considerations make it practically certain that the answers to both Problem B and Problem B′ form sequences which rapidly approach the form of a geometrical progression. Lumping together the electron-produced and photon-produced showers, we may say that the law of distribution of shower size should be essentially

$$P(n;\langle n\rangle)=(\langle n\rangle)^{-1}\{1-(\langle n\rangle)^{-1}\}^{n-1}. \qquad (4')$$

This law may be compared with the usual Poisson law for independent events,

$$P_{\text{ind}}(n;\langle n\rangle)=\{(\langle n\rangle)^n/n!\}e^{-\langle n\rangle}. \qquad (28)$$

The Poisson law is applicable to many distribution problems of physical interest, and Bhabha and Heitler[5] assert that it should hold in the present case. Their argument is based on the assumption that the occurrences of particles in various infinitesimal energy ranges may be regarded as independent events; actually these events are not independent, because some of the particles have produced others, and their chance of doing so depends both upon their original energy and upon the energy lost.

The two distributions (4′) and (28) for $\langle n\rangle=5$ are compared in Fig. 1. It is seen that the fluctuations given by (4′) are large and important; the mean value is not marked by any particular feature of the curve, while the curve given by (28) has a well-defined maximum. In

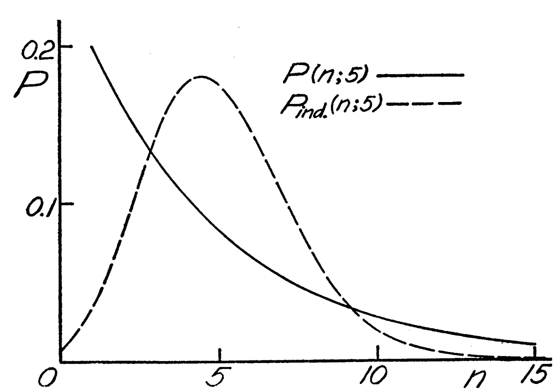

FIG. 1. The distribution law (4′) compared with the Poisson law, for $\langle n\rangle=5$. The functions in question have significance only for integral values of the abscissa, but for convenience continuous curves are plotted.

comparing the probabilities for large showers it is more convenient to consider the quantity

$$S(l; \langle n\rangle) = \sum_{j=l}^{\infty} P(j; \langle n\rangle),$$

which gives the probability for the appearance of l or more electrons. This function, for $\langle n\rangle = 5$, together with the corresponding function $S_{\text{ind}}(l; \langle n\rangle)$ given by (28), is plotted in Fig. 2.

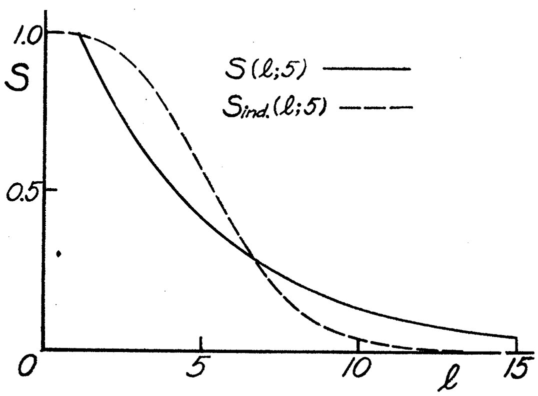

FIG. 2. Total probability for l or more particles according to (4′) and (28), for $\langle n\rangle = 5$. As in Fig. 1, smooth curves are plotted although only discrete points are defined.

The quantities $S_{\text{ind}}(l; \langle n\rangle)$ decrease very rapidly with increasing l; one finds

$$S(15; 5) = 0.044; \quad S_{\text{ind}}(15; 5) = 0.0002.$$
$$S(20; 5) = .014; \quad S_{\text{ind}}(20; 5) = .0000004.$$

The results of some cloud chamber observations on shower sizes were published some time ago by Street and Stevenson.[9] The number of showers observed was rather small, and the results are not corrected for the selective character of the apparatus. Within these limitations, there is qualitative agreement with the type of distribution here suggested. For a thickness of 1.3 cm Pb ($\langle n\rangle \sim 5$), 8 out of 107 showers observed contained more than 10 electrons, and 4 had more than 15 electrons.

A more extended investigation is being carried on in this laboratory by Mr. L. M. Fussell. Results so far obtained, corrected in a reasonable way for the selectivity of the apparatus, are in fairly good agreement with a formula of the type (4′). There seems to be an appreciable excess of simple pairs over the number expected from this expression; a plausible reason for this is indicated below.

[9] J. C. Street and E. C. Stevenson, Phys. Rev. **49**, 425 (1936).

Energy considerations

As explained at the beginning of this section, the most important consequence of our having left out of account the energy losses and energy interchanges of the particles is that the pertinence of the results is strictly limited to the case of thin layers of heavy substances. It is possible to estimate qualitatively the nature of some of the effects which consideration of this neglected aspect of the situation would have on our results:

(1) Unless the primary has quite a high energy—some hundreds of millions of volts—there will simply not be enough energy to produce more than a few particles. Thus unless some way of selecting high energy primaries is used, there will be a preponderance of pairs and small showers over and above that indicated by (4′).

(2) Our neglect of the detailed distribution of energy among the particles made it necessary to use a very crude approximation to the law of photon production by *Bremsstrahlung:* we set the probability that in dt an electron produce a photon "of energy comparable with its own" equal to dt. A much better approximation[4] to the *Bremsstrahlung* formula is obtained if we set as the probability that in dt an electron produce a photon of energy between k and $k+\Delta k$,

$$Pdt\Delta k = dt\Delta k/k. \tag{28a}$$

This makes the probability that in dt an electron produce a photon whose energy equals or exceeds $(1/q)$ of the original energy of the electron equal to $ln\, q \cdot dt$. The full consequences of the more accurate law (28a) could of course be worked out only in a treatment which took account of the detailed energy distribution; two consequences would presumably be roughly as follows:

(a) For a high energy primary, or other particle with many times the energy of an average "low energy" shower particle, $ln\, q$ may be several times our estimated value of 1. Such a particle will thus be unusually effective in producing secondaries. It seems likely that the main effect of this will be to increase the *mean* *number* of particles produced, without changing the *type* of distribution. From our present point of view the high energy primary is to be regarded as equivalent to a number of primaries; the composition of a number of distributions (4′) does not give one of the exact form (4′), but the general behavior is the same.

(b) The mean population of photons accompanying each electron consists not only of the number $\langle m_1 \rangle$ of high energy photons given by (10), but also of a number of lower energy photons, which are capable of producing pairs. These pairs are not able to play the same role as the particles we have been considering, by producing further secondaries, and are usually stopped a short distance beyond the point where they are produced; but an emergent shower will often contain some of them. Having been produced only a short distance back from the point of emergence, and tending to diverge widely because of their low energies, they help to give many showers an appearance of coming from a single center near the bottom of the lead. This effect, which has been of some importance in discouraging the adoption of the multiplicative hypothesis, is also already partly explained by our unmodified formula (4′): The probability of there being 10 or more electrons at $\frac{3}{4}$ cm Pb ($\langle n \rangle \sim 2.25$) is about 0.005; the probability for 10 or more electrons at 1 cm ($\langle n \rangle \sim 3.5$) is about 0.05. Thus any large shower is likely to consist largely of particles from the last millimeter or two of the lead.

Considerations of the fluctuations in number of particles produced have accordingly led us to a formula indicating the general type of distribution of shower sizes to be expected, and have shown how the multiplicative hypothesis can account for two effects—the appearance of large showers from fairly thin pieces of lead, and the apparent divergence of showers from a point—which at first sight might be regarded as strong evidence against it.

III. Distribution in Fractional Energy Loss

The probability that in penetrating dx centimeters of lead an electron of energy E will emit a photon of energy between k and $k+dk$ is

$$Pdxdk = Kdx \cdot I(E/mc^2,\ k/E)dk/k. \qquad (29)$$

The total mean loss by radiation is then

$$dx \cdot \int_0^E Pkdk = KEdx \cdot \int_0^1 Id(k/E). \qquad (30)$$

Here KI is the spectral intensity of the *Bremsstrahlung*. According to the calculations of Bethe and Heitler,[2] KI is practically independent of E/mc^2 for energies above 100 mc^2 (5×10^7 volts). If we choose the two factors so that

$$\int_0^1 Id(k/E) = 1, \qquad (31)$$

then K is numerically equal to 2. Thus by using the variable t, which we introduced without much comment at the beginning of Section II, we have $Kdx = dt$. The mean total rate of energy loss of an electron in lead is then

$$-(dE/dt)_{\text{Av}} = E + \beta, \qquad (32)$$

where $\beta \cong 6 \times 10^6$ volts is the ionization loss per unit thickness. From this equation we see that for a large number of high energy electrons whose energies before and after traversing thickness t are E_0 and E, respectively,

$$[(E+\beta)/(E_0+\beta)]_{\text{Av}} = e^{-t}. \qquad (33)$$

The results of Anderson and Neddermeyer[1] on mean energy loss are in reasonably good agreement with this formula.

The question of fluctuations in energy loss can be approached from either of two points of view:

(1) *The "straggling problem":* The form of the spectrum $I(k/E)$ being given, to determine the probability $P(t;\,E)dE$ that an electron has energy between E and $E+dE$, if before passing through the thickness t it had energy E_0.

(2) *The "spectrum problem":* Given the experimental distribution in energy loss of a large number of electrons, to determine the corresponding form of the spectrum $I(k/E)$.

For the solution of either problem, two approximations seem to be indispensable: First, it is assumed that the form of the spectrum and the total intensity $\int_0^1 Id(k/E)$ are independent of the energy E; second, the ionization loss is neglected. Both of these approximations are justified if the only energies considered are of the order of some tens of millions of volts. They break down, however, when we have to consider particles which are stopped, or even only slowed down to energies of a few million volts. What we have called the "straggling problem" is hence not adapted to any really accurate treatment of straggling in range, but is rather to be regarded simply as the converse of the "spectrum problem." For convenience in dealing with it we introduce a variable

$$\lambda = \ln (E_0/E) \tag{34}$$

and a probability function in the scale of λ:

$$P(t;E)dE = w(t;\lambda)d\lambda = P(t;E)E_0 e^{-\lambda}d\lambda. \tag{35}$$

The "straggling problem" was solved by Bethe and Heitler[10] for the case

$$I(k/E) = -(k/E \ln 2)/\ln (1-k/E). \tag{36}$$

This is a rather rough approximation to the theoretical spectrum curve,[11] giving too much intensity for small (k/E) and too little for larger (k/E)—hence underemphasizing the fluctuations. Using (36), one can verify[10] by using the functional equation,

$$w(t_1+t_2;\lambda)d\lambda = d\lambda \int_0^\lambda w(t_1;\lambda_1)w(t_2;\lambda-\lambda_1)d\lambda_1, \tag{37}$$

that

$$w(t;\lambda)d\lambda = e^{-\lambda}\lambda^{-1+t/\ln 2}d\lambda/\Gamma(t/\ln 2). \tag{38}$$

We now proceed to show how $P(t;E)$ can be found for any spectrum which can be expressed as a power series,

$$I(k/E) = \sum_{j=0}^{\infty} a_j(1-k/E)^{\alpha+j}, \quad \alpha > -1, \tag{39}$$

or as a sum of a number of such series with different values of α. We begin by finding $P(t;E)$ for a spectrum of the form

$$I(k/E) = b_q I_q(k/E) = (b_q k/E)(1-k/E)^q, \quad q > -1. \tag{40}$$

For this spectrum, the probability that in dt an electron's energy changes from ϵ to a value in the range E to $E+dE$ is

$$\begin{aligned} P_{\mathrm{Rad.}}(dt, \epsilon \to (E, E+dE)) &= \{dtdE/(\epsilon - E)\} \cdot b_q I_q(1-E/\epsilon) \\ &= b_q E^q \epsilon^{-q-1} dtdE. \end{aligned} \tag{41}$$

Then $P(t;E)$ satisfies the equation

$$\frac{\partial P}{\partial t} = b_q \int_E^{E_0} E^q \epsilon^{-q-1} P(t;\epsilon)d\epsilon - b_q P(t;E)\int_0^E \epsilon^q E^{-q-1} d\epsilon, \tag{42}$$

which by differentiation reduces to

$$\frac{\partial^2 P}{\partial E \partial t} - qE^{-1}\frac{\partial P}{\partial t} + \{b_q/(q-1)\}\left\{\frac{\partial P}{\partial E} + E^{-1}P\right\} = 0. \tag{43}$$

Introducing the new variable λ by (34), we have:

$$\frac{\partial^2 P}{\partial \lambda \partial t} + q\frac{\partial P}{\partial t} + \{b_q/(q-1)\}\left\{\frac{\partial P}{\partial \lambda} - P\right\} = 0. \tag{44}$$

On making the Laplace transformation,

$$P = \frac{1}{2\pi i}\int_C e^{\lambda z} F(t;z)dz, \tag{45}$$

we see that (44) is satisfied provided

$$(z+q)(\partial F/\partial t) + \{b_q/(q+1)\}(z-1)F = 0. \tag{46}$$

The solution of (42) is then

$$P(t;E) = (1/2\pi i E_0)\int_C \varphi(z) \cdot \exp\left[\lambda z + b_q t\{(z+q)^{-1} - (1+q)^{-1}\}\right]dz, \tag{47}$$

$$P(t;E) = (1/2\pi i E_0)\int_C \varphi(z) \cdot \exp\left[\lambda z + b_q t\{(z+q)^{-1} - (1+q)^{-1}\}\right]dz, \tag{47}$$

where the contour C and the function $\varphi(z)$ may be varied to fit the boundary condition at $t=0$.

Now our general spectrum (39) can be written as a sum of spectra of the form (40):

$$I(k/E) = \sum_{j=0}^{\infty} a_j(1-k/E)^{\alpha+j} = \sum_{j=0}^{\infty} b_{\alpha+j} I_{\alpha+j}(k/E), \tag{48}$$

with

$$b_{\alpha+j} = \sum_{l=0}^{j} a_l. \tag{49}$$

---

[10] Bethe and Heitler, reference 2, p. 101; Heitler, reference 2, p. 225.

[11] The comparison between the theoretical curves and (36) is given in reference 5, Fig. 1. The normalizing factor of (36) relative to the other curves is smaller than ours by about twelve percent. This figure also gives the straight line (28′), with the same difference in normalization.

The essential thing to be remembered at this point is that our fixed spectra assign definite probabilities to given *fractional losses* of energy, independently of the actual value of the energy. The order in which such fractional losses may occur is entirely immaterial. Thus it makes no difference whether the electron is regarded as having passed through thickness t of a substance whose *Bremsstrahlung* spectrum is

$$I_n=\sum_{j=0}^{n} b_{\alpha+j} I_{\alpha+j},$$

or as having passed successively through thicknesses t of substances having the spectra $b_\alpha I_\alpha$, $b_{\alpha+1}I_{\alpha+1}$, $\cdots$, $b_{\alpha+n}I_{\alpha+n}$. We can, accordingly, evaluate the energy distribution by using (47) to define the distributions P_j corresponding to the spectra $b_{\alpha+j}I_{\alpha+j}$ and applying the boundary conditions

$$P_0(0;E)=\delta(E-E_0);\ P_1(0;E)=P_0(t;E);\ \cdots;\ P_n(0;E)=P_{n-1}(t;E). \quad (50)$$

The form of (48) is such that the boundary conditions (50) are easily satisfied. The first of these conditions is satisfied by $\varphi=1$ if C is taken to be a path extending along the imaginary axis from $-i\infty$ to $i\infty$; we agree to deform the path if necessary to insure its passing to the right of the singularity at $-\alpha$. The rest of the boundary conditions are satisfied if we obtain each P_l from P_{l-1} by inserting an additional term in the exponent in the integrand. In the limit $n\rightarrow\infty$, I_n becomes the given spectrum (48), and the desired solution is accordingly

$$P(t;E)=(1/2\pi iE_0)\int_C e^{\lambda z+tf(z)}dz \quad (51)$$

with

$$f(z)=\sum_{j=0}^{\infty} b_{\alpha+j}\{(z+\alpha+j)^{-1}-(1+\alpha+j)^{-1}\}. \quad (52)$$

The series (52) converges for all values of z except $z=-\alpha-j$, provided Σa_j converges, which is true for any function I which remains finite for $(k/E)\rightarrow 0$. If our spectrum is given not as a single series of the form (39), but as a sum of such expressions, we have only to replace $f(z)$ in (51) by the sum of the corresponding expressions of the form of (52).

Equation (51) therefore gives us a complete formal solution of the "straggling problem." For computing actual values of $P(t;E)$ from (51), the only feasible procedure seems to be the saddle-point method.[12] This gives the asymptotic series

$$P(t;E)=At^{-\frac{1}{2}}e^{gt}\{1+\delta_1/t+\delta_2/t^2+\cdots\} \quad (53)$$

with

$$A=(2\pi f''(\zeta))^{-\frac{1}{2}},\quad g=(\lambda/t)\zeta+f(\zeta),$$
$$\delta_1=-(5/24)\{f'''(\zeta)\}^2/\{f''(\zeta)\}^3 + (\tfrac{1}{8})f^{\mathrm{IV}}(\zeta)/\{f''(\zeta)\}^2, \quad (54)$$

and so on; here ζ is the coordinate of the saddle-point, defined by

$$\lambda/t=-f'(\zeta). \quad (55)$$

The quantities (54) are thus all functions of λ/t only.

The distribution (51), (53) has been obtained for the spectrum[11]

$$I(k/E)=1, \quad (28')$$

which corresponds to the approximation (28) mentioned near the end of Section II. This spectrum gives less than the theoretical intensity for small (k/E) and more for larger (k/E), and hence overemphasizes the fluctuations; accordingly we may regard the distribution obtained from (28′) and that obtained from (36) as placing limits on the expected distribution. (28′) is used for some purposes both by Carlson and Oppenheimer[4] and by Bhabha and Heitler.[5]

Using (28′) one finds that[13]

$$f(z)=-\gamma-(d/dt)\ \ln\Gamma(z)=-\gamma-\Psi(z);\quad \gamma=.577216. \quad (56)$$

The $\Psi(z)$ function and its derivatives have been tabulated very fully.[14] The values of A, g, and δ_1 are given in Table VI. $|\delta_2|$ nowhere takes values greater than about 0.005, while $|\delta_3|$ sometimes exceeds 0.01. Hence one uses only the correction δ_1/t, and the results are presumably in error by less than $0.005t^{-2}$.

[12] Courant-Hilbert, *Methoden der Math. Physik*, second edition (Springer, 1931), p. 455.

[13] Whittaker and Watson, *Modern Analysis*, fourth edition, p. 241.

[14] H. T. Davis, *Tables of the Higher Mathematical Functions*, Vols. I and II (The Principia Press. Bloomington, Indiana).

In Fig. 2 distribution curves $E_0P(t;E)$ plotted against E/E_0 are shown for the thickness[15] $t=0.693$ and for three different spectra:

Curve 1, given by (38), for the spectrum (36).
Curve 2, given by Table VI, for the spectrum (28′).
Curve 3, for the spectrum

$$I(k/E)=2(k/E). \qquad (57)$$

Carlson and Oppenheimer at one point in their calculations are obliged to make an approximation which is equivalent to using this spectrum. It badly overemphasizes the fluctuations. For this spectrum, (51) can be evaluated analytically, the result being

$$P(t;E)=e^{-2t}\{\delta(E-E_0)+E_0^{-1}(2t/\lambda)^{\frac{1}{2}}I_1(2[2t\lambda]^{\frac{1}{2}})\}. \qquad (57')$$

For this particular thickness curve 1, which underemphasizes the fluctuations, is simply a straight line; for smaller thicknesses it would show a singularity for $(E/E_0)=1$ and a zero for $(E/E_0)=0$, and *vice versa* for greater thicknesses. Curve 2, which overemphasizes the fluctuations, has a singularity at each end; for $t>1$ it would show a zero at $(E/E_0)=1$. The behavior of curve 2 near $(E/E_0)=1$ is in fact given by

$$E_0P(t;E)\cong(2\pi t)^{-\frac{1}{2}}e^{t(1-\gamma)}(\lambda/t)^{t-1}\cdot(1-1/12t+\cdots),\quad \lambda\ll 1, \qquad (56')$$

as one finds on making suitable approximations to $\Psi(\zeta)$ and its derivatives. In curve 3 the fluctuations are enormously greater than the upper limit given by curve 2; one-fourth of the distribution is contained in a δ-function singularity at $(E/E_0)=1$.

The spectrum problem

For the comparison of limited amounts of energy loss data with theoretical predictions, the use of calculated distribution curves—such as those of Fig. 1—for a few simple cases should suffice; in fact, the data so far published[1] provide only material for comparison with Eq. (33), which gives just the mean loss. It is, however, of some interest to investigate the possibility of using a large amount of very accurate data to give a detailed check on the theory. This can most readily be done by considering the problem already mentioned at the beginning of this section: Given (empirically) the distribution $P(t;E)$, to find the spectrum $I(k/E)$.

This problem can be solved by obtaining a relation between the moments of the energy distribution,

$$\{(E/E_0)^n\}_{\text{Av}}=\int_0^{E_0}(E/E_0)^nP(t;E)dE=\int_0^\infty P(t;E)E_0e^{-(n+1)\lambda}d\lambda, \qquad (58)$$

and those of the spectrum:

$$\mu_n=\int_0^1(1-k/E)^nI(k/E)d(k/E). \qquad (59)$$

By (51) and (58) we obtain

$$\{(E/E_0)^n\}_{\text{Av}}=-(1/2\pi i)\int_C(z-n-1)^{-1}e^{tf(z)}dz. \qquad (60)$$

In all cases where I does not vanish for $(k/E)\to 0$, one can show that $f(z)\to-\infty$ as $z\to\infty$, $|\arg z|\leq\pi/2$. Thus the expression (60) equals just the residue at $z=n+1$:

$$\{(E/E_0)^n\}_{\text{Av}}=e^{tf(n+1)}. \qquad (61)$$

Using (52) and (49), we obtain

$$f(n+1)=\sum_{l=0}^\infty a_l\sum_{j=l}^\infty\{(n+1+\alpha+j)^{-1}-(1+\alpha+j)^{-1}\}=-\sum_{l=0}^\infty a_l\sum_{j=0}^{n-1}(1+\alpha+l+j)^{-1},\quad n\geq 1. \qquad (62)$$

On the other hand one has by (39) and (59):

$$\mu_n=\sum_{l=0}^\infty a_l/(1+\alpha+l+n). \qquad (63)$$

Accordingly,

$$\ln\{(E/E_0)^n\}_{\text{Av}}-\ln\{(E/E_0)^{n+1}\}_{\text{Av}}=t\mu_n. \qquad (64)$$

This solves the problem of obtaining the moments of the spectrum in terms of those of the energy-loss distribution.

[15] This is approximately equal to the thickness (0.35 cm of lead) used by Anderson and Neddermeyer (reference 1); it makes $(E/E_0)_{\text{Av}}=\frac{1}{2}$, and not very many of the particles produce secondary pairs. The error δ_2/t^2 in curve 2 is about one percent.

This same result (64) can be obtained by integrating the diffusion equations for $\{(E/E_0)^n\}_{\text{Av}}$. These may be shown to be

$$
\begin{aligned}
&(d/dt)\{(E/E_0)\}_{\text{Av}}+\{(E/E_0)\}_{\text{Av}}=-(\beta/E_0)\\
&\quad(d/dt)\{(E/E_0)^2\}_{\text{Av}}+(1+\mu_1)\{(E/E_0)^2\}_{\text{Av}}\\
&\qquad\qquad=-2(\beta/E_0)\{(E/E_0)\}_{\text{Av}}\\
&\cdots\cdots\cdots\cdots \qquad (65)\\
&(d/dt)\{(E/E_0)^n\}_{\text{Av}}\\
&\quad+(1+\mu_1+\cdots+\mu_{n-1})\{(E/E_0)^n\}_{\text{Av}}\\
&\qquad\qquad=-n(\beta/E_0)\{(E/E_0)^{n-1}\}_{\text{Av}}.
\end{aligned}
$$

Neglecting the right-hand members, we at once get (64). At first sight one is tempted to suppose that by integrating without this neglect one could give a correct account of the effect of ionization loss. This is not so, for the construction of (65) corresponds to letting particles acquire negative energies after being stopped, instead of discarding them from the distribution (e.g., $(E)_{\text{Av}}\rightarrow-\beta$ for $t\rightarrow\infty$). The problem of allowing correctly for ionization seems to be a very difficult one; it would in any case not be consistent to do so while retaining the approximation of a *Bremsstrahlung* spectrum independent of the energy.

There is apparently no way to obtain the spectrum $I(k/E)$ itself from the values of the μ_n except by direct numerical computation. The most elegant available method seems to be the use of an expansion in Legendre polynomials.

TABLE VI.

λ/t	A	g	δ_1
0.2	1.9980	−1.0878	−0.0848
0.4	1.0035	− .2953	−.0881
0.6	.6736	.1983	−.0912
0.8	.5091	.5746	−.0934
1.0	.4108	.8840	−.0945
1.2	.3451	1.1503	−.0950
1.4	.2990	1.3862	−.0950
1.6	.2643	1.5996	−.0942
1.8	.2367	1.7952	−.0932
2.0	.2151	1.9768	−.0920
2.2	.1973	2.1466	−.0908
2.6	.1698	2.4584	−.0884
3.0	.1496	2.7407	−.0860
3.4	.1340	3.0003	−.0836
3.8	.1217	3.2421	−.0812
4.2	.1116	3.4691	−.0788
4.6	.1032	3.6827	−.0764
5.0	.0962	3.8861	−.0740

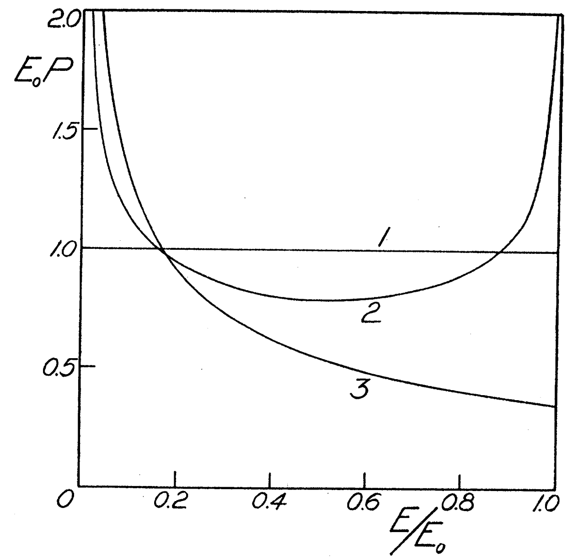

FIG. 3. Distributions in fractional energy loss for $t=0.693$ ($\sim$0.35 cm Pb), calculated for certain simple forms of spectrum (see text). Curve 3 is to be thought of as containing the contribution 0.25 $\delta(E-E_0)$, besides the part shown.

Setting

$$I(k/E)=\sum_{n=0}^{\infty}c_nP_n(1-2k/E), \qquad (66)$$

we have

$$
\begin{aligned}
c_0&=\mu_0\\
c_1&=3(2\mu_1-\mu_0)\\
c_2&=5(6\mu_2-6\mu_1+\mu_0) \qquad (67)\\
c_3&=7(20\mu_3-30\mu_2+12\mu_1-\mu_0)\\
c_4&=9(70\mu_4-140\mu_3+90\mu_2-20\mu_1+\mu_0).\\
&\cdots\cdots\cdots\cdots
\end{aligned}
$$

From the size of the coefficients in (67) it is evident that extremely accurate knowledge of the μ's is required for a reliable determination of the c's; a reasonable procedure in practice would be to calculate the c_n as far as they decrease steadily with increasing n, and neglect the rest. It is this practical difficulty in the use of (64) which gives importance to our analytic solution of the "straggling problem" and the construction of various comparison curves: from a purely formal point of view, the derivation of (64) directly from (65) could be regarded as a solution of both problems at once.

The writer wishes to thank Mr. L. M. Fussell for assistance with the numerical computations.